Languages - **Recognizable, not decidable** - $H_{TM}, A_{TM}, B_{TM}, \overline{E_{TM}}$

Not recognizable - $\overline{H_{TM}}, \overline{A_{TM}}, \overline{B_{TM}}, E_{TM}, D, ALL_{TM}, \overline{ALL_{TM}}$

P - PATH, **NP** - HAMPATH, SS, SAT, 3SAT, 3CNF, CLIQUE, VC, IND

Vertex Cover - $S \subseteq V$ $\forall (x,y) \in E \Rightarrow x \in S$ or $y \in S$ **Independent Set** - $S \subseteq V$ indp set in G if $\forall x,y \in S$ $(x \neq y \Rightarrow (x,y) \notin E$

Clique - $C \subseteq V : \forall x,y \in C,\ x \neq y \Rightarrow \{x,y\} \in E$

$SAT \leq_p 3SAT \leq_p CLIQUE \leq_p IND \leq_p VC$
$\leq_p SS$, HAMPATH

Turing Machine - 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ Q = List of States, Σ = input alphabet $(b \notin \Sigma)$, Γ = tape alphabet $(b \in \Gamma, \Sigma \subseteq \Gamma)$, q_0 = initial state, q_{acc} = accepting state, q_{rej} = rejecting state $(q_{acc} \neq q_{rej})$.

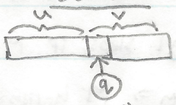
$\delta : (Q - \{q_{acc}, q_{rej}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$

$\mathcal{L}(M)$ $= \{x \in \Sigma^* \mid M \text{ accepts } x\}, L \subseteq \Sigma^*$:

Recognizable - If there exists a TM M s.t. $L = \mathcal{L}(M)$ **Decidable** - if M halts on all inputs.

M checks if $f(\alpha) \in L_2$.
$\mathcal{L}(M) = L_2$

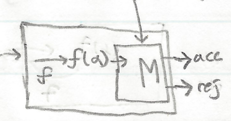
Configuration - uqv $u, v \in \Gamma^*$, $q \in Q$, $v \neq \varepsilon$,



f **Computable** if there is a TM that computes f

"A solution to L_2 yields a soln to L_1"

$\alpha \in L_1$?



M **Computes** f if $\forall x \in \Sigma^*$ M on x halts with output $f(x)$.

$L_1 \leq_m L_2$ L_2 decidable $\Rightarrow L_1$ decidable
L_2 recognizable $\Rightarrow L_1$ recognizable

$L_1 \leq_m L_2$ if there is a Computable function $f : \Sigma^* \to \Sigma^*$ s.t. $\forall \alpha \in \Sigma^*$, $\alpha \in L_1 \Leftrightarrow f(\alpha) \in L_2$.

L decidable $\Leftrightarrow L, \overline{L}$ recognizable

f **running time** of M if $\forall n \in \mathbb{N}$ $f(n) = \max_{\alpha \in \Sigma^*}$ (running time of M on α), **$f \in O(t)$** - M runs in time $O(t)$.

$f \in O(t)$ $\Leftrightarrow \exists c > 0, n_0 \in \mathbb{N} : \forall n > n_0,\ f(n) < c \cdot t(n)$ $\quad O(1) < O(\log n) < O(n) < O(n \log n) < O(n^k) < O(2^n)$

$TIME(n^c)$ $= \{L \mid L$ is a language s.t. for some TM M, $L = \mathcal{L}(M)$ and M runs in $O(n^c)\}$

P $= \bigcup_c TIME(n^c)$, **NP** $= \{L \mid L = \mathcal{L}(M)$ for some polynomial time NTM $M\} = \{L \mid$ There is a polytime verifier for $L\}$

f **Polytime Computable** - There is a TM M that computes f and runs in polynomial time.

NTM - 7 tuple like a TM but $\delta : (Q - \{q_{acc}, q_{rej}\}) \times \Gamma \to P(Q \times \Gamma \times \{L, R\})$

$\mathcal{L}(M) = \{x \in \Sigma^* \mid$ there is some sequence of moves that, on input x, halts and leads to $q_{acc}\}$ NTM M runs in polytime if every possible computation of M on x halts in polytime.

$L_1 \leq_p L_2$ $L_2 \in P \Rightarrow L_1 \in P$
$L_2 \in NP \Rightarrow L_1 \in NP$

Polytime Reducibility - $L_1, L_2 \subseteq \Sigma^*$ $L_1 \leq_p L_2$ if there is a **polytime Computable** function $f : \Sigma^* \to \Sigma^*$ and $\forall x \in \Sigma^*\ x \in L_1 \Leftrightarrow f(x) \in L_2$.

L **NP-Complete** if 1) $L \in NP$ 2) L is **NP-Hard**: $\forall L' \in NP : L' \leq_p L$

$S(n)$ = on input of size n, # squares visited by worktape heads.

$SPACE(S(n))$ $= \{L \mid$ there is a TM M that halts on every input and $\mathcal{L}(M) = L$ and runs in space $O(S(n))\}$

$NSPACE(S(n))$ $= \{L \mid$ there is a NTM M and $\mathcal{L}(M) = L$ and $\exists n_0 \in \mathbb{N}, c \in \mathbb{N} : \forall x \in \Sigma^*, |x| \geq n_0$, every computation of M on x halts and uses $\leq c \cdot S(|x|)$ tape squares$\}$

Savitch's Theorem - for every $f : \mathbb{N} \to \mathbb{R}^+$ s.t. $f(n) \geq \log(n)$, $NSPACE(f(n)) \subseteq SPACE(f^2(n))$

Church-Turing Thesis - Every language that can be recognized/decided by a "algorithm" can be recognized/decided by a Turing Machine.

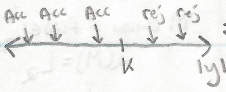
Enumerator - 2-tape TM which accepts no input and prints all strings in L. L recognizable $\Leftrightarrow L$ has an enumerator.

1) <u>Thm</u> - $SPACE(S(n)) \subseteq \bigcup_c TIME(C^{S(n)})$ pf Say M runs in space $d \cdot S(n)$, let $x \in \Sigma^*$, $|x| = n$.

For some C, will show there are $\leq C^{S(n)}$ configurations, and so since M halts, it halts in $\leq C^{S(n)}$ steps.

worktape contents: $|\Gamma|^{S(n)}$ # input head positions: $n+2 < 3^{S(n)}$ $(S(n) \geq \log_2(n))$

worktape head positions: $S(n)$ # of states: $|Q|$ product is $C^{S(n)} = |\Gamma|^{S(n)} \cdot 3^{S(n)} \cdot |Q| \cdot 2$



2) Show $B_{TM} \leq_m \overline{ALL_{TM}}$. Define $f: \Sigma^* \to \Sigma^*$ for input $x = \langle M \rangle$ compute $f(x) = \langle M' \rangle$ where M' on input y runs M on ε for $|y|$ steps, if M accepts ε, M' halts and rejects y, else, M' halts and accepts y.

For $\alpha \in \Sigma^*$, if $\alpha \in B_{TM} \Rightarrow$ M halts after K steps, so M' rejects all strings $|y| \geq K$, so $\langle M' \rangle = f(\alpha) \in \overline{ALL_{TM}}$

$\alpha \notin B_{TM} \Rightarrow$ M never accepts ε so M' accepts all strings, so $\langle M' \rangle = f(\alpha) \notin \overline{ALL_{TM}}$.

$D = \{\langle M \rangle \mid M \text{ a TM and } \langle M \rangle \notin \mathcal{L}(M)\}$ and $\langle M \rangle \in D \Leftrightarrow \langle M \rangle \notin \mathcal{L}(M)$ so $\mathcal{L}(M) \neq D \ \forall M$.

3) Show Clique $\leq_p$ Half Clique. Let $f: \Sigma^* \to \Sigma^*$ s.t. on input $G = \langle G, K \rangle$ let $f(G) = G'$ where G' computed in one of 2 ways: If G has a clique of size K, G' has a clique of size $K = \frac{2K}{2} = \frac{|V'|}{2}$ (graph has 2K vertices now, ie $|V'| = 2K$)

$\boxed{1}$ $K > \frac{|V|}{2}$, add $2K - |V|$ vertices to G, disconnected from rest of graph.

$\boxed{2}$ $K < \frac{|V|}{2}$, add $|V| - 2K$ vertices to G, connected to all other vertices.

If G has a clique of size K, $G' = (V', E')$ has clique of size $|V| - K$, b/c $|V'| = |V| + |V| - 2K = 2(|V| - K) \Rightarrow G'$ has a clique of size $\frac{|V'|}{2} = |V| - K$, so $\langle G' \rangle \in$ Half-Clique, as needed.

$L_1 \leq_p L_2$ $L_2 \in NP \Rightarrow L_1 \in NP$

Let $f: \Sigma^* \to \Sigma^*$ s.t. $\forall \alpha \in \Sigma^*$ $\alpha \in L_1 \Leftrightarrow f(\alpha) \in L_2$ ($f \in O(n^c)$) and $\mathcal{L}(M_1) = L_2$ for NTM M_1 (in time $O(n^d)$)

Consider following NTM which, on input x, computes $f(x)$ and runs M_1 on $f(x)$, accepting and rejecting as M_1 does.

runs in polynomial time: $O(n^{cd})$ since $|f(x)| \in O(n^c)$ and M_2 runs in time $O(n^d)$.